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2009 J. Phys. A: Math. Theor. 42 095208

(http://iopscience.iop.org/1751-8121/42/9/095208)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 095208 (20pp) doi:10.1088/1751-8113/42/9/095208

Reciprocal transformations and local Hamiltonian structures of hydrodynamic-type systems

Simonetta Abenda

Dipartimento di Matematica and CIRAM. Università degli Studi di Bologna, Italy

E-mail: abenda@ciram.unibo.it

Received 15 May 2008, in final form 14 January 2009
Published 5 February 2009
Online at stacks.iop.org/JPhysA/42/095208

Abstract

We start from a hyperbolic Dubrovin and Novikov (DN) hydrodynamic-type system of dimension n which possesses Riemann invariants and we settle the necessary conditions on the conservation laws in the reciprocal transformation so that, after such a transformation of the independent variables, one of the metrics associated with the initial system is flat. We prove the following statement: let $n \geqslant 3$ in the case of reciprocal transformations of a single independent variable or $n \geqslant 5$ in the case of transformations of both the independent variables; then the reciprocal metric may be flat only if the conservation laws in the transformation are linear combinations of the canonical densities of conservation laws, i.e. the Casimirs, the momentum and the Hamiltonian densities associated with the Hamiltonian operator for the initial metric. Then, we restrict ourselves to the case in which the initial metric is either flat or of constant curvature and we classify the reciprocal transformations of one or both the independent variables so that the reciprocal metric is flat. Such characterization has an interesting geometric interpretation: the hypersurfaces of two diagonalizable DN systems of dimension $n \geqslant 5$ are Lie equivalent if and only if the corresponding local Hamiltonian structures are related by a canonical reciprocal transformation.

PACS number: 47.10.Df
Mathematics Subject Classification: 37K05, 37K25, 35F20, 35L60

1. Introduction

Systems of hydrodynamic type are quasilinear evolutionary hyperbolic PDEs of the form

$$u^i_t = \sum_{k=1}^{n} v^i_k(u) u^k_x, \qquad u = (u^1, \ldots, u^n), \qquad u^i_x = \frac{\partial u^i}{\partial x}, \qquad u^i_t = \frac{\partial u^i}{\partial t}. \tag{1}$$

They naturally arise in applications such as gas dynamics, hydrodynamics, chemical kinetics, the Whitham averaging procedure, differential geometry and topological field theory [4, 7, 9, 21, 22]. Dubrovin and Novikov [7] showed that equation (1) is a local Hamiltonian system (DN system) with Hamiltonian $H[u] = \int h(u)\,\mathrm{d}x$, if there exists a flat non-degenerate metric tensor $g(u)$ in $\mathbb{R}^n$ with Christoffel symbols $\Gamma^i_{jk}(u)$, such that the matrix $v^i_k(u)$ can be represented in the form

$$v^i_k(u) = \sum_{k=1}^n \left(g^{il}(u) \frac{\partial^2 h}{\partial u^l \partial u^k}(u) - \sum_{s=1}^n g^{ik}(u)\Gamma^l_{sk}(u)\frac{\partial h}{\partial u^l}(u)\right). \tag{2}$$

In this paper we shall consider DN systems which possess Riemann invariants, i.e. they may be transformed to the diagonal form

$$u^i_t = v^i(\boldsymbol{u})u^i_x, \qquad i = 1, \ldots, n, \tag{3}$$

with $n \geqslant 3$ and with $v^i(\boldsymbol{u})$ all real and distinct (strict hyperbolicity property). We also suppose to work in the space of smooth and rapidly decreasing functions so that $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} f_x = f$.

If $n = 2$, (1) can always be put in a diagonal form and are integrable by the hodograph method. For arbitrary n, Tsarev [21] proved that a DN system as in (1) and (2) can be integrated by a generalized hodograph method only if it may be transformed to the diagonal form. In the latter case, moreover the flat metric is diagonal, the Hamiltonian satisfies

$$\frac{\partial^2 h}{\partial u^i \partial u^j} = \Gamma^i_{ij}(u)\frac{\partial h}{\partial u^i} + \Gamma^j_{ji}(u)\frac{\partial h}{\partial u^j}, \tag{4}$$

and each solution to (4) generates a conserved quantity for the DN system (1), (2) and all Hamiltonian flows generated by these conserved densities pairwise commute. As a consequence, for $n \geqslant 3$, DN systems which possess Riemann invariants are always integrable. We recall that there do also exist DN systems with an infinite number of conserved quantities which do not possess Riemann invariants (see Ferapontov [11] for the classification of the latter when $n = 3$).

Since a non-degenerate flat diagonal metric in $\mathbb{R}^n$ is associated with an orthogonal coordinate system $u^i = u^i(x^1, \ldots, x^n)$, there is a natural link between diagonalizable Hamiltonian systems and n-orthogonal curvilinear coordinates in flat spaces. Upon introducing the Lamé coefficients, which in our case take the form

$$H_i^2(u) = \sum_k \left(\frac{\partial x^i}{\partial u^k}\right)^2,$$

the metric tensor in the coordinate system u^i is diagonal $\mathrm{d}s^2 = \sum_{i=1}^n H_i^2(u)(\mathrm{d}u^i)^2$, and the zero curvature conditions $R_{il,im}(u) = 0$ $(i \neq l \neq m \neq i)$ and $R_{il,il}(u) = 0$ $(i \neq l)$ form an overdetermined system:

$$\frac{\partial^2 H_i}{\partial u^l \partial u^m} = \frac{1}{H_l}\frac{\partial H_l}{\partial u^m}\frac{\partial H_i}{\partial u^l} + \frac{1}{H_m}\frac{\partial H_m}{\partial u^l}\frac{\partial H_i}{\partial u^m}, \tag{5}$$

$$\frac{\partial}{\partial u^l}\frac{\partial H_i}{H_l \partial u^l} + \frac{\partial}{\partial u^i}\frac{\partial H_l}{H_i \partial u^i} + \sum_{m \neq i,l} \frac{1}{H_m^2}\frac{\partial H_i}{\partial u^m}\frac{\partial H_l}{\partial u^m} = 0. \tag{6}$$

Bianchi and Cartan showed that a general solution to the zero curvature equations (5) and (6) can be parametrized locally by $n(n-1)/2$ arbitrary functions of two variables. If the Lamé coefficients $H_i(u)$ are known, one can find $x^i(u^1, \dots, u^n)$ solving the linear overdetermined problem (embedding equations)

$$\frac{\partial^2 x^i}{\partial u^k \partial u^l} = \Gamma^k_{kl}(u)\frac{\partial x^i}{\partial u^k} + \Gamma^l_{lk}(u)\frac{\partial x^i}{\partial u^l}, \qquad \frac{\partial^2 x^i}{\partial (u^l)^2} = \sum_k \Gamma^k_{ll}(u)\frac{\partial x^i}{\partial u^k}. \tag{7}$$

Comparison of equations (4) and (7) implies that the flat coordinates for the metric $g_{ii}(u) = (H^i(u))^2$ are the Casimirs of the corresponding Hamiltonian operator. Finally, Zakharov [24] showed that the dressing method may be used to determine the solutions to the zero curvature equations up to Combescure transformations.

It then follows that the classification of flat diagonal metrics $\mathrm{d}s^2 = g_{ii}(u)(\mathrm{d}u^i)^2$ is an important preliminary step in the classification of integrable Hamiltonian systems of hydrodynamic type. Best known examples of integrable Hamiltonian systems of hydrodynamic type possess Riemann invariants, a pair of compatible flat metrics and have been obtained in the framework of semisimple Frobenius manifolds (axiomatic theory of integrable Hamiltonian systems) [4–6]; in the latter case, one of the flat metrics is also Egorov (i.e. its rotation coefficients are symmetric).

Reciprocal transformations change the independent variables of a system and are an important class of nonlocal transformations which act on hydrodynamic-type systems [1, 2, 12, 13, 19, 20, 23]. Reciprocal transformations map conservation laws to conservation laws and map diagonalizable systems to diagonalizable systems, but act non-trivially on the metrics and on the Hamiltonian structures: for instance, the flatness property and the Egorov property for metrics as well as the locality of the Hamiltonian structure are not preserved, in general, by such transformations. Then, it is natural to investigate under which additional hypotheses the reciprocal system still possesses a local Hamiltonian structure, our ultimate goal being the search for new examples of integrable Hamiltonian systems and the geometrical characterization of the associated hypersurfaces.

Keeping this in mind, in the following we start from a smooth integrable Hamiltonian system in Riemann invariant form

$$u^i_t = v^i(u)u^i_x, \qquad i = 1, \dots, n, \tag{8}$$

with smooth conservation laws

$$B(u)_t = A(u)_x, \qquad N(u)_t = M(u)_x \tag{9}$$

with $B(u)M(u) - A(u)N(u) \neq 0$. In the new independent variables $\hat{x}$ and $\hat{t}$ defined by

$$\mathrm{d}\hat{x} = B(u)\,\mathrm{d}x + A(u)\,\mathrm{d}t, \qquad \mathrm{d}\hat{t} = N(u)\,\mathrm{d}x + M(u)\,\mathrm{d}t, \tag{10}$$

the reciprocal system is still diagonal and takes the form

$$u^i_{\hat{t}} = \frac{B(u)v^i(u) - A(u)}{M(u) - N(u)v^i(u)}u^i_{\hat{x}} = \hat{v}^i(u)u^i_{\hat{x}}. \tag{11}$$

Moreover, the metric of the initial systems $g_{ii}(u)$ transforms to

$$\hat{g}_{ii}(u) = \left(\frac{M(u) - N(u)v^i(u)}{B(u)M(u) - A(u)N(u)}\right)^2 g_{ii}(u), \tag{12}$$

and all conservation laws and commuting flows of the original system (8) may be recalculated in the new independent variables.

If the reciprocal transformation is linear (i.e. A, B, N, M are constant functions), then the reciprocal to a flat metric is still flat and locality and compatibility of the associated Hamiltonian structures are preserved (see [19, 22, 23]).

Under a general reciprocal transformation, the Hamiltonian structure does not behave trivially and a thorough study of reciprocal Hamiltonian structures is still an open problem. Ferapontov and Pavlov [13] constructed the reciprocal Riemannian curvature tensor and the reciprocal Hamiltonian operator when the initial metric is flat, while in [2], we construct the reciprocal Riemannian curvature tensor and the reciprocal Hamiltonian operator when the initial metric is not flat and the initial system also possesses a flat metric.

The classification of the reciprocal Hamiltonian structures is also complicated by the fact that a DN system as in (1) and (2) also possesses an infinite number of nonlocal Hamiltonian structures [12, 15–17]. It is then possible that two DN systems are linked by a reciprocal transformation and that the flat metrics of the first system are not reciprocal to the flat metrics of the second. In [1], we constructed such an example: the genus one modulation (Whitham-CH) equations associated with Camassa–Holm in Riemann invariant form ($n = 3$ in (8)). We proved that the Whitham-CH equations are a DN system and possess a pair of compatible flat metrics (none of the metrics is Egorov). We also proved the connection via a reciprocal transformation of the Whitham-CH equations to the modulation equations associated with the first negative flow of the Korteweg de Vries hierarchy (Whitham-KdV$_{-1}$). In [1], finally we also found the relation between the Poisson structures of the Whitham-KdV$_{-1}$ and the Whitham-CH equations: both systems possess a pair of compatible flat metrics, and the two flat metrics of the first system are respectively reciprocal to the constant curvature and conformally flat metrics of the second (and vice versa).

In view of the above results, in [2] we have started to classify the reciprocal transformations which transform a DN system to another DN system, under the condition that the flat metric tensor $\hat{g}(u)$ of the transformed system is reciprocal to a metric tensor $g(u)$ of the initial system, which is either flat or of constant Riemannian curvature or conformally flat.

In [2], we give necessary and sufficient conditions so that a reciprocal transformation which changes only one independent variable may preserve the flatness of the metric; in particular, we show that the conservation laws in the reciprocal transformation of the independent variable x (respectively t) are linear combinations of Casimirs and momentum densities (respectively Casimirs and Hamiltonian densities).

For an easier comparison with the results known in the literature, we recall that Ferapontov [12] took a reciprocal transformation where the conservation laws in (10) are a linear combination of the Casimirs, momentum and Hamiltonian densities and gave the necessary and sufficient conditions so that starting from a flat metric $g(u)$, the reciprocal metric $\hat{g}(\boldsymbol{u})$ is either a flat or a constant curvature metric. Following Ferapontov [11, 12], we call canonical, a reciprocal transformation in which the integrals in (10) are linear combinations of the $n + 2$ canonical integrals (Casimirs, Hamiltonian and momentum) with respect to the given Hamiltonian structure.

The results in [2, 12] suggest that canonical reciprocal transformations have a privileged role in preserving locality of the Hamiltonian structure. In this paper, we show that canonical transformations are indeed the only reciprocal transformations which may transform the initial metric $g_{ii}(\boldsymbol{u})$ into a reciprocal flat metric $\hat{g}_{ii}(\boldsymbol{u})$ when the dimension of the system is $n \geqslant 3$ (in the case of a transformation of a single independent variable) or $n \geqslant 5$ (in the case of a transformation of both the independent variables).

First of all, in theorems 3.2 and 3.5, we give necessary conditions on the initial metric $g_{ii}(\boldsymbol{u})$ and on the conservation laws (9) in the reciprocal transformation, so that the reciprocal metric (12) is flat. We suppose that the initial system (8) is a DN system which possesses Riemann invariants and we let $g_{ii}(\boldsymbol{u})$ be one of the metrics associated with it. Under such hypotheses, we prove that if the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ in (12) is flat, then the reciprocal transformation is canonical for the initial metric $g_{ii}(\boldsymbol{u})$.

Then, we restrict ourselves to the case in which the initial metric is either flat or of constant curvature and, in theorem 4.1, we classify the reciprocal transformations of one or both the independent variables so that the reciprocal metric is flat. Finally, when both the initial and the transformed metrics are flat, we also discuss the geometric interpretation of the latter theorem in view of the results obtained by Ferapontov in [11]. Indeed, in theorem 4.12, we show that the hypersurfaces of two diagonalizable DN systems are Lie equivalent if and only if the corresponding local Hamiltonian structures are related by a canonical reciprocal transformation which satisfies theorem 4.1.

There are of course still many open problems connected to the classification of local Hamiltonian structures: what about the possible role of other types of transformations among hydrodynamic-type systems? What is the geometrical meaning of the conditions settled by theorems 3.2, 3.5 and 4.1 when the initial metric is not flat? Moreover, there do exist non-diagonalizable integrable Hamiltonian systems; it would be interesting to check whether the same conditions on the conservation laws in the reciprocal transformations preserving the locality of the Hamiltonian structure still hold also in that case.

Finally, several systems of evolutionary PDEs arising in physics may be written as perturbations of hyperbolic systems of PDEs and their classification in the case of Hamiltonian perturbations has recently been started by Dubrovin *et al* [10]. It would also be interesting to investigate the role of reciprocal transformations in this perturbation scheme.

The plan of the paper is as follows. In the following section, we introduce the necessary definitions and we recall some theorems we proved in [2] on the form of the reciprocal Riemannian curvature tensor and of the reciprocal Hamiltonian operator. In section 3, we prove the necessary conditions on the form of the Riemannian curvature tensor and the conservation laws in the reciprocal transformation so that the reciprocal metric is flat. Finally, in section 4, we classify the reciprocal transformation which preserve the flatness of the metric or which transform a constant curvature metric to a flat one and we apply such conditions to some examples.

2. The reciprocal Hamiltonian structure

In this section, we introduce some useful notations, we discuss the role of additive constants in the extended reciprocal transformations and we recall some theorems we proved in [2] which we shall use in the following sections.

We consider a smooth DN Hamiltonian hydrodynamic system in Riemann invariants

$$u^i_t = v^i(\boldsymbol{u})u^i_x, \qquad i = 1, \ldots, n, \tag{13}$$

with $v^i(\boldsymbol{u})$ all real and distinct (strict hyperbolicity property). Let $g_{ii}(\boldsymbol{u})$ be a (covariant) non-degenerate diagonal metric such that for convenient $f^i(u^i)$, $i = 1, \ldots, n$, $g_{ii}(\boldsymbol{u})f^i(u^i)$ is a flat metric associated with the local Hamiltonian operator of the system (13). Let $g^{ii}(\boldsymbol{u}) = 1/g_{ii}(\boldsymbol{u})$. Let $H_i(\boldsymbol{u})$, $\beta_{ij}(\boldsymbol{u})$ and $\Gamma^i_{jk}(\boldsymbol{u})$ respectively be the Lamé coefficients, the rotation coefficients and the Christoffel symbol of a diagonal non-degenerate metric $g_{ii}(\boldsymbol{u})$ associated with (13),

$$H_i(\boldsymbol{u}) = \sqrt{g_{ii}(\boldsymbol{u})}, \qquad \beta_{ij}(\boldsymbol{u}) = \frac{\partial_i H_j(\boldsymbol{u})}{H_i(\boldsymbol{u})}, \qquad i \neq j,$$

$$\Gamma^i_{jk}(\boldsymbol{u}) = \frac{1}{2} g^{im}(\boldsymbol{u}) \left(\frac{\partial g_{mk}(\boldsymbol{u})}{\partial u^j} + \frac{\partial g_{mj}(\boldsymbol{u})}{\partial u^k} - \frac{\partial g_{kj}(\boldsymbol{u})}{\partial u^m} \right).$$

Since the metric is diagonal, the only nonzero Christoffel symbols are

$$\Gamma^j_{ii}(\boldsymbol{u}) = -\frac{H_i(\boldsymbol{u})}{H_j^2(\boldsymbol{u})}\partial_j H_i(\boldsymbol{u}), \qquad \forall\, i \neq j,$$

$$\Gamma^i_{ij}(\boldsymbol{u}) = \frac{\partial_j H_i(\boldsymbol{u})}{H_i(\boldsymbol{u})}, \qquad \forall\, i, j = 1, \ldots, n.$$

Under our hypotheses, the system (13) possesses at least one flat metric. Then, for any other metric associated with (13), the Euler–Darboux equations (6) still hold,

$$\partial_k \beta_{ij}(\boldsymbol{u}) - \beta_{ik}(\boldsymbol{u})\beta_{kj}(\boldsymbol{u}) \equiv 0, \qquad i \neq j \neq k,$$

that is $R^{ij}_{ik}(\boldsymbol{u}) \equiv 0$, $(i \neq j \neq k \neq i)$. For systems (13), Ferapontov [12] constructed the nonlocal Hamiltonian operators $J^{ij}(\boldsymbol{u})$ associated with non-flat metrics $g_{ii}(\boldsymbol{u})$ which take the form

$$J^{ij}(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})\left(\delta^i_j \frac{\mathrm{d}}{\mathrm{d}x} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\right) + \sum_l \epsilon^{(l)} w^i_{(l)}(\boldsymbol{u})u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} w^j_{(l)}(\boldsymbol{u})u^j_x, \tag{14}$$

where $\epsilon^l = \pm 1$, $w^i_{(l)}(\boldsymbol{u})$ are affinors of the metric which satisfy

$$\frac{\partial_j w^i_{(l)}(\boldsymbol{u})}{w^j_{(l)}(\boldsymbol{u}) - w^i_{(l)}(\boldsymbol{u})} = \frac{\partial_j v^i(\boldsymbol{u})}{v^j(\boldsymbol{u}) - v^i(\boldsymbol{u})} = \partial_j \ln H_i(\boldsymbol{u}), \tag{15}$$

and the curvature tensor of the metric takes the form

$$R^{ik}_{ik}(\boldsymbol{u}) = -\frac{\Delta_{ik}(\boldsymbol{u})}{H_i(\boldsymbol{u})H_k(\boldsymbol{u})} \equiv \sum_{(l)} \epsilon^l w^i_{(l)}(\boldsymbol{u}) w^k_{(l)}(\boldsymbol{u}), \qquad i \neq k, \tag{16}$$

where

$$\Delta_{ik}(\boldsymbol{u}) = \partial_i \beta_{ik}(\boldsymbol{u}) + \partial_k \beta_{ki}(\boldsymbol{u}) + \sum_{m \neq i,k} \beta_{mi}(\boldsymbol{u})\beta_{mk}(\boldsymbol{u}).$$

Remark 2.1. In particular, if $g_{ii}(\boldsymbol{u})$ is flat, then $J^{ij}(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})\big(\delta^i_j \frac{\mathrm{d}}{\mathrm{d}x} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\big)$ [7]. If $g_{ii}(\boldsymbol{u})$ is of constant curvature c, then [17]

$$J^{ij}(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})\left(\delta^i_j \frac{\mathrm{d}}{\mathrm{d}x} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\right) + c u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u^j_x. \tag{17}$$

If $g_{ii}(\boldsymbol{u})$ is conformally flat, then

$$R^{ij}_{ij}(\boldsymbol{u}) = w^i(\boldsymbol{u}) + w^j(\boldsymbol{u}), \qquad i \neq j \tag{18}$$

and

$$J^{ij}(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})\left(\delta^i_j \frac{\mathrm{d}}{\mathrm{d}x} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\right) + w^i(\boldsymbol{u})u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u^j_x + u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} w^j(\boldsymbol{u})u^j_x. \tag{19}$$

In the following section, a special role is played by the metrics $g_{ii}(\boldsymbol{u})$ for which the Riemannian curvature tensor takes the special form

$$R^{ik}_{ik}(\boldsymbol{u}) = w^i_{(1)}(\boldsymbol{u}) + w^k_{(1)}(\boldsymbol{u}) + w^i_{(2)}(\boldsymbol{u})v^k(\boldsymbol{u}) + w^k_{(2)}(\boldsymbol{u})v^i(\boldsymbol{u}), \qquad i \neq k, \tag{20}$$

and

$$\begin{aligned} J^{ij}(\boldsymbol{u}) = {} & g^{ii}(\boldsymbol{u})\left(\delta^i_j \frac{\mathrm{d}}{\mathrm{d}x} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\right) + w^i_{(1)}(\boldsymbol{u})u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u^j_x + u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} w^j_{(1)}(\boldsymbol{u})u^j_x \\ & + w^i_{(2)}(\boldsymbol{u})u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} v^j(\boldsymbol{u})u^j_x + v^i(\boldsymbol{u})u^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} w^j_{(2)}(\boldsymbol{u})u^j_x. \end{aligned} \tag{21}$$

Given smooth conservation laws

$$B(\boldsymbol{u})_t = A(\boldsymbol{u})_x, \qquad N(\boldsymbol{u})_t = M(\boldsymbol{u})_x$$

for the system (13), a reciprocal transformation of the independent variables x, t is defined by [20]

$$\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t, \qquad \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t. \tag{22}$$

In [13], Ferapontov and Pavlov have characterized the tensor of the reciprocal Riemannian curvature and the reciprocal Hamiltonian structure when the initial metric $g_{ii}(\boldsymbol{u})$ is flat. In [2], we have computed the Riemannian curvature and the Hamiltonian structure of the reciprocal system

$$u^i_{\hat{t}} = \hat{v}^i(\boldsymbol{u})u^i_{\hat{x}} = \frac{B(\boldsymbol{u})v^i(\boldsymbol{u}) - A(\boldsymbol{u})}{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})}u^i_{\hat{x}}, \tag{23}$$

associated with the reciprocal metric

$$\hat{g}_{ii}(\boldsymbol{u}) = \left(\frac{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}\right)^2 g_{ii}(\boldsymbol{u}), \tag{24}$$

with $g_{ii}(\boldsymbol{u})$ non-flat. In the following, we use the symbols $\hat{H}_i(\boldsymbol{u})$, $\hat{\beta}_{ij}(\boldsymbol{u})$, $\hat{\Gamma}^i_{jk}(\boldsymbol{u})$, $\hat{R}^{ij}_{km}(\boldsymbol{u})$ and $\hat{J}^{ij}$, respectively, for the Lamé coefficients, the rotation coefficients, the Christoffel symbols, the Riemannian curvature tensor and the Hamiltonian operator associated with the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$. To simplify notations, we drop the $\boldsymbol{u}$ dependence in the lengthy formulae.

Theorem 2.2 [2]. *Let $g_{ii}(\boldsymbol{u})$ be the covariant diagonal metric as above for the Hamiltonian system (13) with Riemannian curvature tensor as in (16) or as in (20). Then, for the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ defined in (24), the only possible nonzero components of the reciprocal Riemannian curvature tensor are*

$$\begin{aligned}\hat{R}^{ik}_{ik}(\boldsymbol{u}) = {} & \frac{H_iH_k}{\hat{H}_i\hat{H}_k}R^{ik}_{ik} - (\nabla B)^2 + \frac{H_k}{\hat{H}_k}\nabla^k\nabla_k B + \frac{H_i}{\hat{H}_i}\nabla^i\nabla_i B - \hat{v}^k\hat{v}^i(\nabla N)^2 \\ & + \hat{v}^k\frac{H_i}{\hat{H}_i}\nabla^i\nabla_i N + \hat{v}^i\frac{H_k}{\hat{H}_k}\nabla^k\nabla_k N - (\hat{v}^k + \hat{v}^i)\langle\nabla B, \nabla N\rangle, \qquad i \neq k\end{aligned} \tag{25}$$

where

$$\begin{aligned}\langle\nabla B(\boldsymbol{u}), \nabla N(\boldsymbol{u})\rangle &= \sum_m g^{mm}(\boldsymbol{u})\partial_m B(\boldsymbol{u})\partial_m N(\boldsymbol{u}), \\ \nabla^i\nabla_i B(\boldsymbol{u}) &= g^{ii}(\boldsymbol{u})\left(\partial_i^2 B(\boldsymbol{u}) - \sum_m \Gamma^m_{ii}(\boldsymbol{u})\partial_m B(\boldsymbol{u})\right), \\ \nabla^i\nabla_j B(\boldsymbol{u}) &= g^{ii}(\boldsymbol{u})\big(\partial_i\partial_j B(\boldsymbol{u}) - \Gamma^i_{ij}(\boldsymbol{u})\partial_i B(\boldsymbol{u}) - \Gamma^j_{ij}(\boldsymbol{u})\partial_j B(\boldsymbol{u})\big).\end{aligned}$$

In [2], we computed the reciprocal affinors and the reciprocal Hamiltonian operator of a hydrodynamic system (13) with (nonlocal) Hamiltonian operator (14). At this aim, we introduce the auxiliary flows

$$\begin{aligned}&u^i_\tau = n^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_j N(\boldsymbol{u}), \qquad u^i_\zeta = b^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_j B(\boldsymbol{u}), \\ &u^i_{t_{(l)}} = w^i_{(l)}(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_j H^{(l)}(\boldsymbol{u}),\end{aligned} \tag{26}$$

respectively, generated by the densities of conservation laws associated with the reciprocal transformation (22), $B(\boldsymbol{u})$, $N(\boldsymbol{u})$, and by the densities of conservation laws $H^{(l)}(\boldsymbol{u})$ associated with the affinors $w^i_{(l)}(\boldsymbol{u})$ of the Riemannian curvature tensor (16). By construction, all the auxiliary flows commute with (13). Introducing the following closed form

$$\begin{cases} \mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t + P(\boldsymbol{u})\,\mathrm{d}\tau + Q(\boldsymbol{u})\,\mathrm{d}\zeta + \sum_l T^{(l)}(\boldsymbol{u})\,\mathrm{d}t_{(l)}, \\ \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t + R(\boldsymbol{u})\,\mathrm{d}\tau + S(\boldsymbol{u})\,\mathrm{d}\zeta + \sum_l Z^{(l)}(\boldsymbol{u})\,\mathrm{d}t_{(l)}, \\ \mathrm{d}\hat{\tau} = \mathrm{d}\tau, \qquad \mathrm{d}\hat{\zeta} = \mathrm{d}\zeta, \qquad \mathrm{d}\hat{t}_{(l)} = \mathrm{d}t_{(l)}, \end{cases} \tag{27}$$

it is easy to verify that the reciprocal auxiliary flows

$$u^i_{\hat{\tau}} = \hat{n}^i(\boldsymbol{u})u^i_{\hat{x}}, \qquad u^i_{\hat{\zeta}} = \hat{b}^i(\boldsymbol{u})u^i_{\hat{x}}, \qquad u^i_{\hat{t}^{(l)}} = \hat{w}^i_{(l)}(\boldsymbol{u})u^i_{\hat{x}},$$

satisfy

$$\begin{aligned} \hat{n}^i(\boldsymbol{u}) &= n^i B - P + (Nn^i - R)\hat{v}^i = \frac{H_i}{\hat{H}_i}n^i - P - \hat{v}^i R, \\ \hat{b}^i(\boldsymbol{u}) &= b^i B - Q + (Nb^i - S)\hat{v}^i = \frac{H_i}{\hat{H}_i}b^i - Q - \hat{v}^i S, \\ \hat{w}^i_{(l)}(\boldsymbol{u}) &= w^i_{(l)} B - T^{(l)} + \big(Nw^i_{(l)} - Z^{(l)}\big)\hat{v}^i = \frac{H_i}{\hat{H}_i}w^i_{(l)} - T^{(l)} - \hat{v}^i Z^{(l)}. \end{aligned} \tag{28}$$

Using (27), we immediately conclude that $T^{(l)}(\boldsymbol{u})$, $Z^{(l)}(\boldsymbol{u})$ satisfy

$$n^i(\boldsymbol{u}) = \nabla^i\nabla_i N + \sum_{(l)} \epsilon_{(l)} Z^{(l)} w^i_{(l)}, \qquad b^i(\boldsymbol{u}) = \nabla^i\nabla_i B + \sum_{(l)} \epsilon_{(l)} T^{(l)} w^i_{(l)}. \tag{29}$$

Moreover, we have

$$\begin{aligned} v^i(\boldsymbol{u}) &= \frac{\partial_i A(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i M(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, & w^i_{(l)}(\boldsymbol{u}) &= \frac{\partial_i T^{(l)}(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i Z^{(l)}(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, \\ b^i(\boldsymbol{u}) &= \frac{\partial_i Q(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i S(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, & n^i(\boldsymbol{u}) &= \frac{\partial_i P(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i R(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}. \end{aligned} \tag{30}$$

Using (29) and (30), $Q(\boldsymbol{u})$, $R(\boldsymbol{u})$ and $P(\boldsymbol{u}) + S(\boldsymbol{u})$ are uniquely defined (up to additive constants) by the following identities:

$$\begin{aligned} &Q(\boldsymbol{u}) = \frac{1}{2}(\nabla B)^2 + \frac{1}{2}\sum_l \epsilon_{(l)}(T^{(l)})^2, \qquad R(\boldsymbol{u}) = \frac{1}{2}(\nabla N)^2 + \frac{1}{2}\sum_l \epsilon_{(l)}(Z^{(l)})^2, \\ &P(\boldsymbol{u}) + S(\boldsymbol{u}) = \langle \nabla N, \nabla B\rangle + \sum_l \epsilon_{(l)} T^{(l)} Z^{(l)}. \end{aligned} \tag{31}$$

If the Riemannian curvature tensor associated with $g_{ii}(\boldsymbol{u})$ takes the special form (20), then (29) take the special form

$$\begin{aligned} n^i(\boldsymbol{u}) &= \nabla^i\nabla_i N + w^i_{(1)}N + Z^{(1)} + w^i_{(2)}M + v^i Z^{(2)}, \\ b^i(\boldsymbol{u}) &= \nabla^i\nabla_i B + w^i_{(1)}B + T^{(1)} + w^i_{(2)}A + v^i T^{(2)}, \end{aligned} \tag{32}$$

and

$$\begin{aligned} Q(\boldsymbol{u}) &= \tfrac{1}{2}(\nabla B)^2(\boldsymbol{u}) + B(\boldsymbol{u})T^{(1)}(\boldsymbol{u}) + A(\boldsymbol{u})T^{(2)}(\boldsymbol{u}), \\ R(\boldsymbol{u}) &= \tfrac{1}{2}(\nabla N)^2(\boldsymbol{u}) + N(\boldsymbol{u})Z^{(1)}(\boldsymbol{u}) + M(\boldsymbol{u})Z^{(2)}(\boldsymbol{u}), \\ P(\boldsymbol{u}) + S(\boldsymbol{u}) &= \langle \nabla N, \nabla B\rangle + T^{(1)}N + T^{(2)}M + Z^{(1)}B + Z^{(2)}A. \end{aligned} \tag{33}$$

Remark 2.3. The addition of constants to the rhs of (31) leave invariant the reciprocal transformation in the sense that the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$, the reciprocal Riemannian tensor $\hat{R}^{ij}_{ij}(\boldsymbol{u})$, the reciprocal Hamiltonian operator $\hat{J}^{ij}(\boldsymbol{u})$ and the reciprocal Hamiltonian velocity flow $\hat{v}^i(\boldsymbol{u})$ are not effected by them. These constants just effect the auxiliary flows. Indeed, let $Q(\boldsymbol{u})$, $P(\boldsymbol{u})$, $R(\boldsymbol{u})$ and $S(\boldsymbol{u})$ be as in (31) and let us consider the modified closed form

$$\begin{cases} \mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t + (P(\boldsymbol{u}) + \alpha)\,\mathrm{d}\tau + (Q(\boldsymbol{u}) + \beta)\,\mathrm{d}\zeta + \sum_l T^{(l)}(\boldsymbol{u})\,\mathrm{d}t_{(l)}, \\ \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t + (R(\boldsymbol{u}) + \gamma)\,\mathrm{d}\tau + (S(\boldsymbol{u}) + \delta)\,\mathrm{d}\zeta + \sum_l Z^{(l)}(\boldsymbol{u})\,\mathrm{d}t_{(l)}, \\ \mathrm{d}\hat{\tau} = \mathrm{d}\tau, \qquad \mathrm{d}\hat{\zeta} = \mathrm{d}\zeta, \end{cases}$$

with $\alpha, \beta, \gamma, \delta$ arbitrary constants

$$\hat{n}^i_m(\boldsymbol{u}) = \hat{n}^i(\boldsymbol{u}) - \beta - \delta\hat{v}^i(\boldsymbol{u}), \qquad \hat{b}^i_m(\boldsymbol{u}) = \hat{b}^i(\boldsymbol{u}) - \alpha - \gamma\hat{v}^i(\boldsymbol{u}),$$

with $\hat{n}^i(\boldsymbol{u})$ and $\hat{b}^i(\boldsymbol{u})$ as in(28).

The following alternative expressions for the reciprocal Riemann curvature tensor and the reciprocal Hamiltonian structure hold.

Theorem 2.4. *Let $g_{ii}(\boldsymbol{u})$ be the metric for the Hamiltonian hydrodynamic system (13), with Riemannian curvature tensor as in (16). Then, after the reciprocal transformation (22), the nonzero components of the reciprocal Riemannian curvature tensor are*

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \sum_l \epsilon^{(l)}\hat{w}^i_{(l)}(\boldsymbol{u})\hat{w}^k_{(l)}(\boldsymbol{u}) + \hat{v}^i(\boldsymbol{u})\hat{n}^k(\boldsymbol{u}) + \hat{v}^k(\boldsymbol{u})\hat{n}^i(\boldsymbol{u}) + \hat{b}^i(\boldsymbol{u}) + \hat{b}^k(\boldsymbol{u}), \quad i \neq k, \quad (34)$$

where the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ and the reciprocal affinors $\hat{n}^i(\boldsymbol{u})$, $\hat{b}^i(\boldsymbol{u})$ and $\hat{w}^i_{(l)}(\boldsymbol{u})$ are as in (24) and (28), respectively, with $Q(\boldsymbol{u})$, $P(\boldsymbol{u})$, $R(\boldsymbol{u})$ and $S(\boldsymbol{u})$ as in (31).

Let $g_{ii}(\boldsymbol{u})$ be the metric for the Hamiltonian hydrodynamic system (13), with a Riemannian curvature tensor as in (20), then the nonzero components of the transformed curvature tensor take the form

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \hat{n}^i(\boldsymbol{u})\hat{v}^k(\boldsymbol{u}) + \hat{n}^k(\boldsymbol{u})\hat{v}^i(\boldsymbol{u}) + \hat{b}^i(\boldsymbol{u}) + \hat{b}^k(\boldsymbol{u}), \qquad i \neq k, \qquad (35)$$

where the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ and the reciprocal affinors $\hat{n}^i(\boldsymbol{u})$, $\hat{b}^i(\boldsymbol{u})$ and $\hat{w}^i_{(l)}(\boldsymbol{u})$ are as in (24) and (28), respectively, with $Q(\boldsymbol{u})$, $P(\boldsymbol{u})$, $R(\boldsymbol{u})$ and $S(\boldsymbol{u})$ as in (33).

Formula (34) has already been proven in [2]. To prove (35), it is sufficient to insert (32) and (33) into (25).

Corollary 2.5. *Let the reciprocal transformation change only x ($N(\boldsymbol{u}) = 0$ and $M(\boldsymbol{u}) = 1$ in (22)), then the nonzero components of the transformed curvature tensor take the form*

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = B^2(\boldsymbol{u})R^{ik}_{ik}(\boldsymbol{u}) + B(\boldsymbol{u})(\nabla^i\nabla_i B(\boldsymbol{u}) + \nabla^k\nabla_k B(\boldsymbol{u})) - (\nabla B(\boldsymbol{u}))^2. \qquad (36)$$

Moreover, if the Riemannian curvature tensor of $g_{ii}(\boldsymbol{u})$ takes the form as in (16), then

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \sum_l \epsilon^{(l)}\hat{w}^i_{(l)}(\boldsymbol{u})\hat{w}^k_{(l)}(\boldsymbol{u}) + \hat{b}^i(\boldsymbol{u}) + \hat{b}^k(\boldsymbol{u}),$$

if the Riemannian curvature tensor associated with $g_{ii}(\boldsymbol{u})$ takes the form (20) then the nonzero components of the transformed curvature tensor take the form

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \hat{w}^i_{(2)}(\boldsymbol{u})\hat{v}^k_{(l)}(\boldsymbol{u}) + \hat{w}^k_{(2)}(\boldsymbol{u})\hat{v}^i_{(l)}(\boldsymbol{u}) + \hat{b}^i(\boldsymbol{u}) + \hat{b}^k(\boldsymbol{u}). \qquad (37)$$

If the reciprocal transformation changes only t ($B(\boldsymbol{u}) = 1$ and $A(\boldsymbol{u}) = 0$ in (22)), then the nonzero components of the transformed curvature tensor satisfy

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \frac{M^2R^{ik}_{ik} + M(v^k\nabla^i\nabla_i N + v^i\nabla^k\nabla_k N) - v^iv^k(\nabla N)^2}{(M - Nv^i)(M - Nv^k)}. \qquad (38)$$

Moreover, if the Riemannian curvature tensor of $g_{ii}(\boldsymbol{u})$ takes the form as in (16), then

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \sum_l \epsilon^{(l)}\hat{w}^i_{(l)}(\boldsymbol{u})\hat{w}^k_{(l)}(\boldsymbol{u}) + \hat{v}^i(\boldsymbol{u})\hat{n}^k(\boldsymbol{u}) + \hat{v}^k(\boldsymbol{u})\hat{n}^i(\boldsymbol{u}),$$

if the Riemannian curvature tensor associated with $g_{ii}(\boldsymbol{u})$ takes the form (20), then the nonzero components of the transformed curvature tensor take the form

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \hat{w}^i_{(1)}(\boldsymbol{u}) + \hat{w}^k_{(1)}(\boldsymbol{u}) + \hat{v}^i(\boldsymbol{u})\hat{n}^k(\boldsymbol{u}) + \hat{v}^k(\boldsymbol{u})\hat{n}^i(\boldsymbol{u}). \tag{39}$$

Formulae (36), (38) and their expressions when $R^{ik}_{ik}(\boldsymbol{u})$ is as in (16) have already been proven in [2]. To prove (37) (respectively (39)) it is sufficient to insert (32) and (33) into (36) (respectively (38)).

3. Necessary conditions for reciprocal flat metrics

In this section, we start from an integrable Hamiltonian system $u^i_t = v^i(\boldsymbol{u})u^i_x$, $i = 1, \ldots, n$ and we investigate the necessary conditions on the initial metric and on the conservation laws in the reciprocal transformation so that the reciprocal metric is flat. The conditions settled by theorem 3.5 on the conservation laws in the reciprocal transformations are very strict: if $n \geqslant 5$, they must be linear combinations with constant coefficients of the Casimirs, the momentum and the Hamiltonian densities with respect to the initial Hamiltonian structure. The same theorem settles also very strict conditions on the admissible form of the Riemannian curvature tensor associated with the initial metric $g_{ii}(\boldsymbol{u})$. In the case of reciprocal transformations of a single independent variable the necessary conditions are even more restrictive: if $n \geqslant 3$, the conservation law is a linear combination of Casimirs and momentum densities (respectively of Casimirs and Hamiltonian densities) if just the x variable (respectively the t variable) changes.

Definition 3.1. *Following Ferapontov [11, 12], we call canonical, a reciprocal transformation as in (22), in which the integrals, up to additive constants, are linear combinations of the canonical integrals (Casimirs, Hamiltonian and momentum) with respect to the given Hamiltonian structure.*

Remark 3.1. If the initial metric $g_{ii}(\boldsymbol{u})$ is not flat, a Casimir density (respectively a momentum density, a Hamiltonian density) associated with the corresponding nonlocal Hamiltonian operator $J^{ij}(\boldsymbol{u})$ in (14) is a conservation law $h(u)$ such that $J^{ij}\partial_j h(\boldsymbol{u}) \equiv 0$ (respectively $J^{ij}\partial_j h(\boldsymbol{u}) \equiv u^i_x$, $J^{ij}\partial_j h(\boldsymbol{u}) \equiv v^i(\boldsymbol{u})u^i_x$). We remark that, under the hypotheses of the following theorem, for each Hamiltonian structure with k nonlocalities in the Hamiltonian operator, there do exist $(n+k+2)$ canonical integrals as proven by Maltsev and Novikov [18].

In the following theorem, we settle the necessary conditions for reciprocal flat metrics in the case of a transformation of a single variable.

Theorem 3.2. *Let $u^i_t = v^i(\boldsymbol{u})u^i_x$, $i = 1, \ldots, n$, $n \geqslant 3$, be an integrable strictly hyperbolic DN hydrodynamic-type system as in (13), let $g_{ii}(\boldsymbol{u})$ be one of its metrics with Hamiltonian operator $J^{ij}(\boldsymbol{u})$ as in (14).*

(i) Let $\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t$, $\mathrm{d}\hat{t} = \mathrm{d}t$, be a reciprocal transformation such that the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ defined in (24) is flat.

Then $B(\boldsymbol{u})$ is a linear combination of the Casimirs and the momentum densities (up to an additive constant), and $g_{ii}(\boldsymbol{u})$ is either a flat or a constant curvature or a conformally flat metric.

(ii) Let $\mathrm{d}\hat{x} = \mathrm{d}x, \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t$*, be a reciprocal transformation such that the reciprocal metric* $\hat{g}_{ii}(\boldsymbol{u})$ *defined in (24) is flat. In the case* $n = 3$*, let moreover* $v^i(\boldsymbol{u}) \not\equiv 0, i = 1, \ldots, 3$.

Then $N(\boldsymbol{u})$ *is a linear combination of the Casimirs and the Hamiltonian densities (up to an additive constant), and the Riemannian curvature tensor associated with the initial metric* $g_{ii}(\boldsymbol{u})$ *takes the form*

$$R^{ij}_{ij}(\boldsymbol{u}) = w^i(\boldsymbol{u})v^j(\boldsymbol{u}) + w^j(\boldsymbol{u})v^i(\boldsymbol{u}), \qquad i \neq j, \tag{40}$$

with $w^i(\boldsymbol{u})$ *(possibly null) affinors.*

Proof. To compute the form of the Riemannian curvature tensor associated with the initial metric $g_{ii}(\boldsymbol{u})$ it is sufficient to invert the reciprocal transformation (42) and to apply theorem 2.4 to the reciprocal flat metric $\hat{g}_{ii}(\boldsymbol{u})$.

(i) If the reciprocal transformation changes only x ($N(\boldsymbol{u}) \equiv 0$, $M(\boldsymbol{u}) \equiv 1$) and the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat, the Riemann curvature tensor associated with the initial metric $g_{ii}(\boldsymbol{u})$ takes the form $R^{ik}_{ik}(\boldsymbol{u}) = w^i_{(1)}(\boldsymbol{u}) + w^k_{(1)}(\boldsymbol{u})$ $(i \neq k)$, with possibly constant or null affinors $w^i_{(1)}(\boldsymbol{u})$ (see [13]). According to corollary (2.5), the zero curvature equations $\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \hat{b}^i(\boldsymbol{u}) + \hat{b}^k(\boldsymbol{u}) \equiv 0$ $(i \neq k)$, for the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ are then equivalent to

$$0 \equiv \hat{b}^i(\boldsymbol{u}) = B(\boldsymbol{u})b^i(\boldsymbol{u}) - Q(\boldsymbol{u}), \qquad i = 1, \ldots, n,$$

as follows from (37) with $Q(\boldsymbol{u})$ as in (33). Since $b^i(\boldsymbol{u}) = \frac{\partial_i Q(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})}$ $(i = 1, \ldots, n)$, we immediately conclude that there exists a constant κ such that

$$u^i_{\hat{\zeta}} \equiv b^i(\boldsymbol{u})u^i_x \equiv J^{ij}(\boldsymbol{u})\partial_j B(\boldsymbol{u}) = \kappa u^i_x, \qquad i = 1, \ldots, n,$$

that is $B(\boldsymbol{u})$ is a linear combination of the Casimirs and the momentum densities up to an additive constant.

(ii) Similarly, if the reciprocal transformation changes only t ($B(\boldsymbol{u}) \equiv 1$, $A(\boldsymbol{u}) \equiv 0$) and the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat, the Riemann curvature tensor associated with the initial metric $g_{ii}(\boldsymbol{u})$ takes the form $R^{ik}_{ik}(\boldsymbol{u}) = w^i_{(2)}(\boldsymbol{u})v^k(\boldsymbol{u}) + w^k_{(2)}(\boldsymbol{u})v^i(\boldsymbol{u})$, $(i \neq k)$, with possibly constant or null affinors $w^i_{(2)}(\boldsymbol{u})$ (see [13]). According to corollary (2.5), the zero curvature equations for the reciprocal metric, $\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \hat{v}^i(\boldsymbol{u})\hat{n}^k(\boldsymbol{u}) + \hat{v}^k(\boldsymbol{u})\hat{n}^i(\boldsymbol{u}) \equiv 0$, $(i \neq k)$, are equivalent to

$$0 \equiv \hat{n}^i(\boldsymbol{u}) = \frac{M(\boldsymbol{u})n^i(\boldsymbol{u}) - R(\boldsymbol{u})v^i(\boldsymbol{u})}{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})}, \qquad i = 1, \ldots, n. \tag{41}$$

Since $v^i(\boldsymbol{u}) = \frac{\partial_i M(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, n^i(\boldsymbol{u}) = \frac{\partial_i R(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}$, $(i = 1, \ldots, n)$, we immediately conclude that there exists a constant κ such that

$$u^i_{\hat{\tau}} \equiv n^i(\boldsymbol{u})u^i_x \equiv J^{ij}(\boldsymbol{u})\partial_j N(\boldsymbol{u}) = \kappa v^i(\boldsymbol{u})u^i_x, \qquad i = 1, \ldots, n,$$

that is the density of conservation law associated with the inverse transformation is a linear combination of the Casimirs and the Hamiltonian densities up to an additive constant. □

Remark 3.3. Theorem 3.2 is not applicable in the case $n = 2$. For instance, in the case of a transformation of the single variable x, we get the zero curvature condition $\hat{b}^1(\boldsymbol{u}) = -\hat{b}^2(\boldsymbol{u})$ and it is possible to construct non-canonical reciprocal transformations which preserve the flatness of the metric. Here is a counterexample suggested by the second referee: let us take a linear two-component system

$$u^1_t = pu^1_x, \qquad u^2_t = qu^2_x,$$

where p, q are constants. It has infinitely many Hamiltonian structures, let us take the one corresponding to the metric $g = (\mathrm{d}u^1)^2 + (\mathrm{d}u^2)^2$. Let us consider a reciprocal transformation of x only, $\mathrm{d}\hat{x} = B(u^1, u^2)\,\mathrm{d}x + A(u^1, u^2)\,\mathrm{d}t, \hat{t} = t$. For the above system, the general form of a density of conservation law is $B(u^1, u^2) = f^1(u^1) + f^2(u^2)$. Let us require that the transformed metric be flat: this gives a functional–differential equation for f^1 and f^2 which can be solved explicitly.

In particular, if $B(u^1, u^2) = a + bu^1 + cu^2 + \frac{d}{2}((u^1)^2 + (u^2)^2)$, then the flatness condition gives $b^2+c^2 = 2ad$. This is the case of canonical integrals discussed in theorem 3.2. However, there is another solution:

$$B(u^1, u^2) = a\exp(u^1) + b\exp(-u^1) + c\sin(u^2) + d\cos(u^2)$$

with $c^2 + d^2 = 4ab$. Thus, the reciprocal metric is flat, although the density B is not a linear combination of canonical integrals.

Remark 3.4. In the case of time transformations and $n = 3$, the hypothesis $v^i(\boldsymbol{u}) \not\equiv 0$ ensures $\hat{v}^i(\boldsymbol{u}) \not\equiv 0$. If $n = 3$ and $v^3(\boldsymbol{u}) = 0$, then theorem 3.2 is not applicable for transformations of the independent variable t. Indeed, the zero curvature equations for the transformed metric take the form

$$\hat{v}^3(\boldsymbol{u}) = 0, \qquad \hat{n}^3(\boldsymbol{u}) = 0, \qquad \hat{n}^2(\boldsymbol{u})\hat{v}^1(\boldsymbol{u}) + \hat{n}^1(\boldsymbol{u})\hat{v}^2(\boldsymbol{u}) \equiv 0,$$

instead of (41). The condition $\hat{n}^3(\boldsymbol{u}) \equiv 0$ implies $n^3(\boldsymbol{u}) \equiv 0$, but we cannot conclude that $\hat{n}^1(\boldsymbol{u}) = 0 = \hat{n}^2(\boldsymbol{u})$ and in general we may get a transformed flat metric with non-canonical transformations. Indeed, let

$$u^1_t = 2u^1_x, \qquad u^2_t = u^2_x, \qquad u^3_t = 0.$$

The above system is integrable and possesses a local Hamiltonian structure associated with the flat metric $g = (\mathrm{d}u^1)^2 + (\mathrm{d}u^2)^2 + (\mathrm{d}u^3)^2$. Let the reciprocal transformation be $\mathrm{d}\hat{x} = \mathrm{d}x, \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t$, with

$$N(\boldsymbol{u}) = \exp(u^1) + \exp(-u^1) + 2\sqrt{2}\cos\left(\frac{u^2}{2}\right) + 2\sin\left(\frac{u^2}{2}\right) + u^3,$$

$$M(\boldsymbol{u}) = \exp(u^1) + \exp(-u^1) + 4\sqrt{2}\cos\left(\frac{u^2}{2}\right) + 4\sin\left(\frac{u^2}{2}\right).$$

Then the zero curvature equations for the transformed metric $\hat{g}_{ii}(\boldsymbol{u})$ are identically satisfied and

$$n^1(\boldsymbol{u}) = \exp(u^1) + \exp(-u^1), \qquad n^2(\boldsymbol{u}) = -\frac{\sqrt{2}}{2}\cos\left(\frac{u^2}{2}\right) - \frac{1}{2}\sin\left(\frac{u^2}{2}\right), \qquad n^3(\boldsymbol{u}) = 0.$$

In the following theorem, we settle the necessary conditions for reciprocal flat metrics in the case of a reciprocal transformation of both the independent variables.

Theorem 3.5. *Let $u^i_t = v^i(\boldsymbol{u})u^i_x, i = 1, \ldots, n, n \geqslant 5$, be an integrable strictly hyperbolic DN hydrodynamic-type system as in (13), let $g_{ii}(\boldsymbol{u})$ be one of its metrics with Hamiltonian operator $J^{ij}(\boldsymbol{u})$ as in (14). Let*

$$\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t, \qquad \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t, \tag{42}$$

be a reciprocal transformation such that the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ defined in (24) is flat. Then *(i) There exist (possibly null) affinors* $w^i_{(l)}(\boldsymbol{u}), i = 1, \ldots, n, l = 1, 2,$ *such that the Riemannian curvature tensor of the initial metric* $g_{ii}(\boldsymbol{u})$ *takes the form*

$$R^{ij}_{ij}(\boldsymbol{u}) = w^i_{(1)}(\boldsymbol{u}) + w^j_{(1)}(\boldsymbol{u}) + w^i_{(2)}(\boldsymbol{u})v^j(\boldsymbol{u}) + w^j_{(2)}(\boldsymbol{u})v^i(\boldsymbol{u}), \qquad i \neq j. \tag{43}$$

(ii) The reciprocal transformation (42) is canonical with respect to $J^{ij}(\boldsymbol{u})$*, the Hamiltonian operator associated with the initial metric* $g_{ii}(\boldsymbol{u})$*. In particular, the auxiliary flows*

$$u^i_\zeta = b^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_j B(\boldsymbol{u}), \qquad u^i_\tau = n^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_j N(\boldsymbol{u}),$$

associated with such transformations are linear combinations of the x and t flows.

Proof. To verify property (i) it is sufficient to invert the reciprocal transformation (42) and to apply theorem 2.4 to the reciprocal flat metric $\hat{g}_{ii}(\boldsymbol{u})$.

We now prove statement (ii) in the case of a general reciprocal transformation (42) and let the initial metric $g_{ii}(\boldsymbol{u})$ have Riemann curvature tensor as in (43).

Let $n = 5$. The zero curvature equations associated with the reciprocal flat metric $\hat{g}_{ii}(\boldsymbol{u})$ are

$$\hat{b}^i(\boldsymbol{u}) + \hat{b}^j(\boldsymbol{u}) + \hat{n}^i(\boldsymbol{u})\hat{v}^j(\boldsymbol{u}) + \hat{n}^j(\boldsymbol{u})\hat{v}^i(\boldsymbol{u}) = 0, \qquad i \neq j.$$

Using the strict hyperbolicity hypothesis, it is elementary to show that they may be equivalently expressed as

$$\hat{b}^i(\boldsymbol{u}) = -\hat{n}^1(\boldsymbol{u})\hat{v}^i(\boldsymbol{u}), \qquad \hat{n}^i(\boldsymbol{u}) = \hat{n}^1(\boldsymbol{u}), \qquad i = 1, \ldots, 5.$$

For $n \geqslant 5$, it is also easy to prove by induction that the system of zero curvature equations in the $2n$ variables $\hat{b}^i(\boldsymbol{u}), \hat{n}^i(\boldsymbol{u})$ has rank $2n - 1$ and that

$$\hat{b}^i(\boldsymbol{u}) = -\hat{n}^1(\boldsymbol{u})\hat{v}^i(\boldsymbol{u}), \qquad \hat{n}^i(\boldsymbol{u}) = \hat{n}^1(\boldsymbol{u}), \qquad i = 1, \ldots, n. \tag{44}$$

Since $\hat{n}^j(\boldsymbol{u})$ are affinors of the transformed metric $\hat{g}_{ii}(\boldsymbol{u})$, using (15) for the transformed metric and (44), we have $\partial_k\hat{n}^j(\boldsymbol{u}) \equiv 0, k \neq j$. Using again (44), we then conclude that there exists a (possibly null) constant κ_0 such that

$$\hat{b}^i(\boldsymbol{u}) = -\kappa_0\hat{v}^i(\boldsymbol{u}), \qquad \hat{n}^i(\boldsymbol{u}) = \kappa_0, \qquad i = 1, \ldots, n. \tag{45}$$

For the inverse reciprocal transformation, we have

$$\begin{aligned} \mathrm{d}x &= \hat{B}(\boldsymbol{u})\,\mathrm{d}\hat{x} + \hat{A}(\boldsymbol{u})\,\mathrm{d}\hat{t} + \hat{Q}(\boldsymbol{u})\,\mathrm{d}\hat{\zeta} + \hat{P}(\boldsymbol{u})\,\mathrm{d}\hat{\tau}, \\ \mathrm{d}t &= \hat{N}(\boldsymbol{u})\,\mathrm{d}\hat{x} + \hat{M}(\boldsymbol{u})\,\mathrm{d}\hat{t} + \hat{S}(\boldsymbol{u})\,\mathrm{d}\hat{\zeta} + \hat{R}(\boldsymbol{u})\,\mathrm{d}\hat{\tau}, \qquad \zeta = \hat{\zeta}, \qquad \tau = \hat{\tau}, \end{aligned}$$

with

$$\begin{aligned} \hat{B}(\boldsymbol{u}) &= \frac{M(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}, & \hat{A}(\boldsymbol{u}) &= -\frac{A(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}, \\ \hat{N}(\boldsymbol{u}) &= -\frac{N(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}, & \hat{M}(\boldsymbol{u}) &= \frac{B(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}, \\ \hat{Q}(\boldsymbol{u}) &= \frac{S(\boldsymbol{u})A(\boldsymbol{u}) - Q(\boldsymbol{u})M(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}, & \hat{S}(\boldsymbol{u}) &= \frac{Q(\boldsymbol{u})N(\boldsymbol{u}) - S(\boldsymbol{u})B(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}, \\ \hat{P}(\boldsymbol{u}) &= \frac{R(\boldsymbol{u})A(\boldsymbol{u}) - P(\boldsymbol{u})M(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}, & \hat{R}(\boldsymbol{u}) &= \frac{P(\boldsymbol{u})N(\boldsymbol{u}) - R(\boldsymbol{u})B(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}. \end{aligned} \tag{46}$$

Since

$$\hat{v}^i(\boldsymbol{u}) = \frac{B(\boldsymbol{u})v^i(\boldsymbol{u}) - A(\boldsymbol{u})}{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})} = \frac{\partial_i\hat{A}(\boldsymbol{u})}{\partial_i\hat{B}(\boldsymbol{u})} = \frac{\partial_i\hat{M}(\boldsymbol{u})}{\partial_i\hat{N}(\boldsymbol{u})}, \qquad i = 1, \ldots, n,$$

and the reciprocal affinors satisfy ($i = 1, \dots, n$)

$$\hat{n}^i(\boldsymbol{u}) = B(\boldsymbol{u})n^i(\boldsymbol{u}) - P(\boldsymbol{u}) + (N(\boldsymbol{u})n^i(\boldsymbol{u}) - R(\boldsymbol{u}))\hat{v}^i(\boldsymbol{u}) = \frac{\partial_i \hat{P}(\boldsymbol{u})}{\partial_i \hat{B}(\boldsymbol{u})} = \frac{\partial_i \hat{R}(\boldsymbol{u})}{\partial_i \hat{N}(\boldsymbol{u})},$$

$$\hat{b}^i(\boldsymbol{u}) = B(\boldsymbol{u})n^i(\boldsymbol{u}) - Q(\boldsymbol{u}) + (N(\boldsymbol{u})n^i(\boldsymbol{u}) - S(\boldsymbol{u}))\hat{v}^i(\boldsymbol{u}) = \frac{\partial_i \hat{Q}(\boldsymbol{u})}{\partial_i \hat{B}(\boldsymbol{u})} = \frac{\partial_i \hat{S}(\boldsymbol{u})}{\partial_i \hat{N}(\boldsymbol{u})},$$

we immediately conclude that there exist constants $\kappa_1, \dots, \kappa_4$ such that

$$\hat{S}(\boldsymbol{u}) = -\kappa_0 \hat{M}(\boldsymbol{u}) - \kappa_1, \qquad \hat{Q}(\boldsymbol{u}) = -\kappa_0 \hat{A}(\boldsymbol{u}) - \kappa_2,$$
$$\hat{R}(\boldsymbol{u}) = \kappa_0 \hat{N}(\boldsymbol{u}) - \kappa_3, \qquad \hat{P}(\boldsymbol{u}) = \kappa_0 \hat{B}(\boldsymbol{u}) - \kappa_4.$$

Inserting (46) into the above equations, we then get

$$Q(\boldsymbol{u}) = \kappa_2 B(\boldsymbol{u}) + \kappa_1 A(\boldsymbol{u}), \qquad S(\boldsymbol{u}) = \kappa_2 N(\boldsymbol{u}) + \kappa_1 M(\boldsymbol{u}) + \kappa_0,$$
$$P(\boldsymbol{u}) = \kappa_4 B(\boldsymbol{u}) + \kappa_3 A(\boldsymbol{u}) - \kappa_0, \qquad R(\boldsymbol{u}) = \kappa_4 N(\boldsymbol{u}) + \kappa_3 M(\boldsymbol{u}),$$

from which the assertion follows. □

Remark 3.6. If $n = 4$, the system of the six zero curvature equations for the transformed metric $\hat{g}_{ii}(\boldsymbol{u})$ has maximal rank 6 in the unknowns $\hat{b}^i, \hat{n}^i$, and it is possible to express, say $\hat{b}^i(\boldsymbol{u}), \hat{n}^i(\boldsymbol{u}), i = 2, 3, 4$ in function of $\hat{b}^1(\boldsymbol{u})$ and $\hat{n}^1(\boldsymbol{u})$. Moreover the condition $\hat{n}^i(\boldsymbol{u}) = \hat{n}^1(\boldsymbol{u}), i = 2, 3, 4$ is satisfied if and only if $\hat{b}^1(\boldsymbol{u}) = -\hat{v}^1(\boldsymbol{u})\hat{n}^1(\boldsymbol{u})$.

The above observation implies that, for $n = 4$, there exist non-canonical transformations which preserve the flatness of the metric when $\hat{n}^i(\boldsymbol{u}) \neq \hat{n}^1(\boldsymbol{u})$ for $i \in \{2, 3, 4\}$.

4. Classification of the reciprocal transformations which preserve the flatness of the metric or transform constant curvature metrics to flat metrics

Theorems 3.2 and 3.5 state that only the reciprocal transformations which are canonical with respect to the initial Hamiltonian structure may transform the initial metric to a flat one, respectively for $n \geqslant 3$ (reciprocal transformations of a single independent variable) or $n \geqslant 5$ (reciprocal transformations of both the independent variables). In view of the above, in this section we restrict ourselves to the case in which the initial metric $g_{ii}(\boldsymbol{u})$ is either flat ($w^i_{(1)} \equiv 0 \equiv w^i_{(2)}, i = 1, \dots, n$, in (43)) or of constant curvature $2c$ ($w^i_{(1)} \equiv c$, $w^i_{(2)} \equiv 0, i = 1, \dots, n$, in (43)). Then, in theorem 4.1, we completely characterize which reciprocal transformations map $g_{ii}(\boldsymbol{u})$ to flat metric $\hat{g}_{ii}(\boldsymbol{u})$.

Finally, the case in which both the initial and the transformed Hamiltonian structure are local has also a nice geometric interpretation in view of the results by Ferapontov [11], which we present in theorem 4.12.

Theorem 4.1. *Let $n \geqslant 5$ and let $u^i_t = v^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_j H(\boldsymbol{u}), i = 1, \dots, n$, be a DN integrable strictly hyperbolic hydrodynamic-type system, with $J^{ij}(\boldsymbol{u})$ being the Hamiltonian operator associated with the metric $g_{ii}(\boldsymbol{u})$ and $H(\boldsymbol{u})$ the Hamiltonian density. Let $\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t$, $\mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t$ be a reciprocal transformation with $A(\boldsymbol{u}), B(\boldsymbol{u}), M(\boldsymbol{u})$ and $N(\boldsymbol{u})$ not all constant functions.*

(A) Let the metric $g_{ii}(\boldsymbol{u})$ be flat. Then the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ defined in (24) is flat if and only if one of the following alternatives hold:

(A.1) there exist constants $\kappa_1 \neq 0, \kappa_2, \kappa_3$ such that

$$M(\boldsymbol{u}) = \kappa_1, \qquad N(\boldsymbol{u}) = \kappa_2, \qquad (\nabla B)^2(\boldsymbol{u}) = \kappa_3 (\kappa_1 B(\boldsymbol{u}) - \kappa_2 A(\boldsymbol{u}));$$

(A.2) there exist constants $\kappa_1 \neq 0, \kappa_2, \kappa_3$ *such that*

$$B(\boldsymbol{u}) = \kappa_1, \qquad A(\boldsymbol{u}) = \kappa_2, \qquad (\nabla N)^2(\boldsymbol{u}) = \kappa_3\,(\kappa_1 M(\boldsymbol{u}) - \kappa_2 N(\boldsymbol{u}))\,;$$

(A.3) there exist constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ *such that*

$$(\nabla B)^2(\boldsymbol{u}) = 2\kappa_1 A(\boldsymbol{u}) + 2\kappa_2 B(\boldsymbol{u}), \qquad (\nabla N)^2(\boldsymbol{u}) = 2\kappa_3 M(\boldsymbol{u}) + 2\kappa_4 N(\boldsymbol{u}),$$
$$\langle \nabla B(\boldsymbol{u}), \nabla N(\boldsymbol{u})\rangle = \kappa_1 M(\boldsymbol{u}) + \kappa_2 N(\boldsymbol{u}) + \kappa_3 A(\boldsymbol{u}) + \kappa_4 B(\boldsymbol{u}).$$

(B) Let the metric $g_{ii}(\boldsymbol{u})$ *be of constant curvature* $2c$*. Then the reciprocal metric* $\hat{g}_{ii}(\boldsymbol{u})$ *defined in (24) is flat if and only if one of the following alternatives hold:*

(B.1) there exist constants $\kappa_1 \neq 0, \kappa_3$*, such that*

$$M(\boldsymbol{u}) = \kappa_1, \qquad N(\boldsymbol{u}) \equiv 0, \qquad (\nabla B)^2(\boldsymbol{u}) + 2cB^2(\boldsymbol{u}) = 2\kappa_3 B(\boldsymbol{u});$$

(B.2) there exist constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ *such that*

$$(\nabla B)^2(\boldsymbol{u}) + 2cB^2(\boldsymbol{u}) = 2\kappa_1 A(\boldsymbol{u}) + 2\kappa_2 B(\boldsymbol{u}),$$
$$(\nabla N)^2(\boldsymbol{u}) + 2cN^2(\boldsymbol{u}) = 2\kappa_3 M(\boldsymbol{u}) + 2\kappa_4 N(\boldsymbol{u}),$$
$$\langle \nabla B(\boldsymbol{u}), \nabla N(\boldsymbol{u})\rangle + 2cB(\boldsymbol{u})N(\boldsymbol{u}) = \kappa_1 N(\boldsymbol{u}) + \kappa_2 M(\boldsymbol{u}) + \kappa_3 A(\boldsymbol{u}) + \kappa_4 B(\boldsymbol{u}).$$

Remark 4.2. Case (A.i) (respectively (A.ii)) includes the reciprocal transformations of the single variable x (respectively the single variable t) when $\kappa_1 = 1, \kappa_2 = 0$.

Case (B.i) corresponds to reciprocal transformations of the single variable x (note that only $N(\boldsymbol{u}) \equiv 0$ is admissible if $c \neq 0$). Finally, it is not possible to transform a constant curvature metric to a flat one by a transformation of the single variable t.

Proof. Let $g_{ii}(\boldsymbol{u})$ be either a flat ($c = 0$) or a constant curvature metric ($c \neq 0$).

We prove first (A.i) and (B.i). Let $\kappa_1 \neq 0, \kappa_2$ be constants such that

$$M(\boldsymbol{u}) \equiv \kappa_1, \qquad N(\boldsymbol{u}) \equiv \kappa_2.$$

Then, the only possibly nonzero elements of the reciprocal Riemannian curvature tensor take the form

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = 2c\frac{H_i(\boldsymbol{u})H_k(\boldsymbol{u})}{\hat{H}_i(\boldsymbol{u})\hat{H}_k(\boldsymbol{u})} - (\nabla B)^2(\boldsymbol{u}) + \frac{H_i(\boldsymbol{u})}{\hat{H}_i(\boldsymbol{u})}\nabla^i\nabla_i B(\boldsymbol{u}) + \frac{H_k(\boldsymbol{u})}{\hat{H}_k(\boldsymbol{u})}\nabla^k\nabla_k B(\boldsymbol{u}),$$

where

$$\hat{H}_i(\boldsymbol{u}) = \frac{\kappa_1 - \kappa_2 v^i(\boldsymbol{u})}{B(\boldsymbol{u})\kappa_1 - A(\boldsymbol{u})\kappa_2} H_i(\boldsymbol{u}), \qquad \hat{v}^i(\boldsymbol{u}) = \frac{B(\boldsymbol{u})v^i(\boldsymbol{u}) - A(\boldsymbol{u})}{\kappa_1 - \kappa_2 v^i(\boldsymbol{u})}, \qquad i = 1, \ldots, n.$$

From the necessary condition found in theorems 3.2 and 3.5,

$$b^i(\boldsymbol{u}) \equiv \nabla^i\nabla_i B(\boldsymbol{u}) + 2cB(\boldsymbol{u}) = \kappa_3 + \kappa_4 v^i(\boldsymbol{u}), \qquad i = 1, \ldots, n, \tag{47}$$

we infer

$$(\nabla B)^2(\boldsymbol{u}) + 2cB^2(\boldsymbol{u}) = 2\kappa_3 B(\boldsymbol{u}) + 2\kappa_4 A(\boldsymbol{u}) + \kappa_5. \tag{48}$$

If we insert (47) and (48) inside the expression of the transformed Riemannian curvature tensor, we immediately get

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = -\kappa_5 + (\kappa_1\kappa_4 + \kappa_3\kappa_2)(\hat{v}^i(\boldsymbol{u}) + \hat{v}^k(\boldsymbol{u})) + 2c\kappa_2^2\hat{v}^i(\boldsymbol{u})\hat{v}^k(\boldsymbol{u}).$$

Then the condition $\hat{R}^{ik}_{ik}(\boldsymbol{u}) \equiv 0$, is equivalent to either

$$c = \kappa_5 = \kappa_1\kappa_4 + \kappa_3\kappa_2 = 0,$$

or to

$$c \neq 0, \qquad \text{and} \qquad \kappa_5 = \kappa_2 = \kappa_4 = 0,$$

from which cases (A.i) and (B.i) immediately follow.

We now prove (A.ii). Let $\kappa_1 \neq 0, \kappa_2$ be constants such that $B(\boldsymbol{u}) \equiv \kappa_1$, $A(\boldsymbol{u}) \equiv \kappa_2$ and let the initial metric $g_{ii}(\boldsymbol{u})$ be flat. Then, the only possibly nonzero elements of the reciprocal Riemannian curvature tensor take the form,

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \frac{H^i(\boldsymbol{u})}{\hat{H}^i(\boldsymbol{u})}\nabla^i\nabla_i N(\boldsymbol{u})\hat{v}^k(\boldsymbol{u}) + \frac{H^k(\boldsymbol{u})}{\hat{H}^k(\boldsymbol{u})}\nabla^k\nabla_k N(\boldsymbol{u})\hat{v}^i(\boldsymbol{u}) - \hat{v}^i(\boldsymbol{u})\hat{v}^k(\boldsymbol{u})\,(\nabla N)^2(\boldsymbol{u}), \qquad i \neq k.$$

Inserting into the above equation ($i = 1, \dots, n$)

$$\hat{H}_i(\boldsymbol{u}) = \frac{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})}{\kappa_1 M(\boldsymbol{u}) - \kappa_2(\boldsymbol{u})N(\boldsymbol{u})} H_i(\boldsymbol{u}), \qquad \hat{v}^i(\boldsymbol{u}) = \frac{\kappa_1 v^i(\boldsymbol{u}) - \kappa_2}{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})},$$

we get

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \hat{v}^i(\boldsymbol{u})\hat{v}^k(\boldsymbol{u})\left(\frac{(\kappa_1 M(\boldsymbol{u}) - \kappa_2 N(\boldsymbol{u})n^i(\boldsymbol{u})}{\kappa_1 v^i(\boldsymbol{u}) - \kappa_2} + \frac{(\kappa_1 M(\boldsymbol{u}) - \kappa_2 N(\boldsymbol{u})n^k(\boldsymbol{u})}{\kappa_1 v^k(\boldsymbol{u}) - \kappa_2} - (\nabla N)^2\right).$$

Since $\hat{v}^i(\boldsymbol{u}) \not\equiv 0$, the conditions $\hat{R}^{ik}(\boldsymbol{u}) \equiv 0$, $(i \neq k)$ are equivalent to either $(\nabla N)^2(\boldsymbol{u}) \equiv 0$ or

$$\frac{\partial_i\,(\nabla N)^2(\boldsymbol{u})}{(\nabla N)^2(\boldsymbol{u})} = \frac{\partial_i\,(\kappa_1 M(\boldsymbol{u}) - \kappa_2 N(\boldsymbol{u}))}{(\kappa_1 M(\boldsymbol{u}) - \kappa_2 N(\boldsymbol{u}))}, \qquad i = 1, \dots, n,$$

from which case (A.ii) immediately follows. In particular, under the same hypotheses, we also have

$$n^i(\boldsymbol{u}) = \kappa_3(\kappa_1 v^i(\boldsymbol{u}) + \kappa_2),$$
$$\hat{n}^i(\boldsymbol{u}) = -\frac{1}{2}(\nabla N)^2(\boldsymbol{u})\hat{v}^i(\boldsymbol{u}) + n^i(\boldsymbol{u})\frac{\kappa_1 M(\boldsymbol{u}) - \kappa_2 N(\boldsymbol{u})}{M(\boldsymbol{u} - N(\boldsymbol{u})v^i(\boldsymbol{u})} \equiv 0.$$

If $\kappa_1 \neq 0, \kappa_2$ are constants such that $B(\boldsymbol{u}) \equiv \kappa_1$, $A(\boldsymbol{u}) \equiv \kappa_2$ and the initial metric $g_{ii}(\boldsymbol{u})$ is of constant curvature $c \neq 0$, then it is easy to show that the transformed metric $\hat{g}_{ii}$ cannot be flat.

To prove (A.iii) and (B.ii), we use the closed form

$$\begin{cases} \mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t + P(\boldsymbol{u})\,\mathrm{d}\tau + Q(\boldsymbol{u})\,\mathrm{d}\zeta, \\ \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t + R(\boldsymbol{u})\,\mathrm{d}\tau + S(\boldsymbol{u})\,\mathrm{d}\zeta, \\ \mathrm{d}\hat{\tau} = \mathrm{d}\tau, \qquad \mathrm{d}\hat{\zeta} = \mathrm{d}\zeta, \end{cases} \tag{49}$$

associated with the auxiliary flows

$$\begin{aligned} u^i_\tau &= n^i(\boldsymbol{u})u^i_x = (\nabla^i\nabla_i N(\boldsymbol{u}) + 2cN(\boldsymbol{u}))u^i_x, \\ u^i_\zeta &= b^i(\boldsymbol{u})u^i_x = (\nabla^i\nabla_i B(\boldsymbol{u}) + 2cB(\boldsymbol{u}))u^i_x. \end{aligned} \tag{50}$$

In view of the results of the previous section, the auxiliary flows (50) are necessarily linear combinations of the x and t flows. We impose that the conservation laws in the reciprocal transformation satisfy the necessary conditions settled in theorem 3.5. Then there exist constants κ_j, $j = 1, \dots, 8$ such that

$$b^i(\boldsymbol{u}) = \kappa_1 v^i(\boldsymbol{u}) + \kappa_2, \qquad (\nabla B)^2(\boldsymbol{u}) + 2cB(\boldsymbol{u}) = 2\kappa_1 A(\boldsymbol{u}) + 2\kappa_2 B(\boldsymbol{u}) + 2\kappa_6,$$
$$n^i(\boldsymbol{u}) = \kappa_3 v^i(\boldsymbol{u}) + \kappa_4, \qquad (\nabla N)^2(\boldsymbol{u}) + 2cN(\boldsymbol{u}) = 2\kappa_3 M(\boldsymbol{u}) + 2\kappa_4 N(\boldsymbol{u}) + 2\kappa_5,$$
$$P(\boldsymbol{u}) = \kappa_3 A(\boldsymbol{u}) + \kappa_4 B(\boldsymbol{u}) + \kappa_7, \qquad S(\boldsymbol{u}) = \kappa_1 M(\boldsymbol{u}) + \kappa_2 N(\boldsymbol{u}) + \kappa_8,$$
$$\langle \nabla B, \nabla N\rangle + 2cBN \equiv P + S = \kappa_1 M(\boldsymbol{u}) + \kappa_2 N(\boldsymbol{u}) + \kappa_3 A(\boldsymbol{u}) + \kappa_4 B(\boldsymbol{u}) + \kappa_7 + \kappa_8.$$

If we insert the above expressions into the right-hand side of (28) we get

$$\hat{n}^i(\boldsymbol{u}) = -\kappa_7 - \kappa_5\hat{v}^i(\boldsymbol{u}), \qquad \hat{b}^i(\boldsymbol{u}) = -\kappa_6 - \kappa_8\hat{v}^i(\boldsymbol{u}).$$

Finally, the elements of the Riemannian curvature tensor are

$$\begin{aligned}\hat{R}^{ik}_{ik}(\boldsymbol{u}) &= \hat{n}^i(\boldsymbol{u})\hat{v}^i(\boldsymbol{u}) + \hat{n}^k(\boldsymbol{u})\hat{v}^k(\boldsymbol{u}) + \hat{b}^i(\boldsymbol{u}) + \hat{b}^k(\boldsymbol{u}) \\ &= -2\kappa_6 - (\kappa_7 + \kappa_8)(\hat{v}^i(\boldsymbol{u}) + \hat{v}^k(\boldsymbol{u})) - 2\kappa_5\hat{v}^i(\boldsymbol{u})\hat{v}^k(\boldsymbol{u}), \qquad i \neq k,\end{aligned}$$

so that $\hat{R}^{ik}_{ik}(\boldsymbol{u}) \equiv 0$ if and only if $\kappa_5 = \kappa_6 = \kappa_7 + \kappa_8 = 0$, and assertions (A.iii) and (B.ii) easily follow. □

Example 4.3. If $B(\boldsymbol{u})$ and $N(\boldsymbol{u})$ are non-trivial independent Casimirs of the flat metric $g_{ii}(\boldsymbol{u})$ and $(\nabla B(\boldsymbol{u}))^2 \neq 0$, then there exist a constant α and $A(\boldsymbol{u})$ such that, under the reciprocal transformation $\mathrm{d}\hat{x} = (\alpha B(\boldsymbol{u}) + N(\boldsymbol{u}))\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t$, the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u}) = g_{ii}(\boldsymbol{u})/(\alpha B(\boldsymbol{u}) + N(\boldsymbol{u}))^2$ is flat.

Example 4.4. If $B(\boldsymbol{u})$ is a density of momentum for the flat metric $g^{ii}(\boldsymbol{u})$ and $(\nabla B(\boldsymbol{u}))^2 - 2B(\boldsymbol{u}) = 2\alpha$, then under the reciprocal transformation $\mathrm{d}\hat{x} = (B(\boldsymbol{u}) + \alpha)\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t$, the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u}) = g_{ii}(\boldsymbol{u})/(B(\boldsymbol{u}) + \alpha)^2$ is flat.

Example 4.5. If $B(\boldsymbol{u})$ and $N(\boldsymbol{u})$ are non-trivial independent Casimirs of the flat metric $g_{ii}(\boldsymbol{u})$ and $(\nabla N(\boldsymbol{u}))^2 \neq 0$, then there exist a constant α and $M(\boldsymbol{u})$ such that, under the reciprocal transformation $\mathrm{d}\hat{t} = (\alpha N(\boldsymbol{u}) + B(\boldsymbol{u}))\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t$, the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat.

Example 4.6. If $N(\boldsymbol{u})$ is a density of Hamiltonian for the flat metric $g_{ii}(\boldsymbol{u})$ and $(\nabla N(\boldsymbol{u}))^2 - 2M(\boldsymbol{u}) = 2\alpha$, then under the reciprocal transformation $\mathrm{d}\hat{t} = N(\boldsymbol{u})dx + (M(\boldsymbol{u}) + \alpha)\,\mathrm{d}t$, the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat.

Example 4.7. Let $N(\boldsymbol{u})$ be a density of momentum and let $B(\boldsymbol{u})$ be a density of Hamiltonian for the flat metric $g_{ii}(\boldsymbol{u})$. Then under the reciprocal transformation

$$\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + \tfrac{1}{2}(\nabla B)^2(\boldsymbol{u})\,\mathrm{d}t, \qquad \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t,$$

such that $(\nabla N)^2(\boldsymbol{u}) = 2N(\boldsymbol{u})$, $\langle\nabla N(\boldsymbol{u}), \nabla B(\boldsymbol{u})\rangle = N(\boldsymbol{u}) + B(\boldsymbol{u})$, the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat.

Example 4.8. Let $N(\boldsymbol{u}) = M(\boldsymbol{u}) = 1$ and let $B(\boldsymbol{u})$ be a density of Hamiltonian for the flat metric $g_{ii}(\boldsymbol{u})$. Then under the reciprocal transformation

$$\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + \tfrac{1}{2}(\nabla B)^2(\boldsymbol{u})\,\mathrm{d}t, \qquad \mathrm{d}\hat{t} = \mathrm{d}x + \mathrm{d}t,$$

the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat.

Example 4.9. Let $N(\boldsymbol{u})$ be a density of momentum and let $B(\boldsymbol{u})$ be a density of Hamiltonian for the metric $g_{ii}(\boldsymbol{u})$ with constant curvature $2c$. Then under the reciprocal transformation

$$\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + \left(\tfrac{1}{2}(\nabla B)^2(\boldsymbol{u}) + cB^2(\boldsymbol{u})\right)\mathrm{d}t, \qquad \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + M(\boldsymbol{u})\,\mathrm{d}t,$$

such that $(\nabla N)^2(\boldsymbol{u}) + 2cN^2(\boldsymbol{u}) - 2N(\boldsymbol{u}) \equiv 0$, $\langle\nabla N(\boldsymbol{u}), \nabla B(\boldsymbol{u})\rangle + 2cN(\boldsymbol{u})B(\boldsymbol{u}) - N(\boldsymbol{u}) - B(\boldsymbol{u}) \equiv 0$, the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat.

Example 4.10. Let $N(\boldsymbol{u})$ be a density of Hamiltonian and let $B(\boldsymbol{u})$ be a Casimir for the metric $g_{ii}(\boldsymbol{u})$ with constant curvature $2c$. Then under the reciprocal transformation

$$\mathrm{d}\hat{x} = B(\boldsymbol{u})\,\mathrm{d}x + A(\boldsymbol{u})\,\mathrm{d}t, \qquad \mathrm{d}\hat{t} = N(\boldsymbol{u})\,\mathrm{d}x + \left(\tfrac{1}{2}(\nabla N)^2(\boldsymbol{u}) + cN^2(\boldsymbol{u})\right)\mathrm{d}t,$$

such that $(\nabla B)^2(\boldsymbol{u}) + 2cB^2(\boldsymbol{u}) \equiv 0$, $\langle\nabla N(\boldsymbol{u}), \nabla B(\boldsymbol{u})\rangle + 2cN(\boldsymbol{u})B(\boldsymbol{u}) - B(\boldsymbol{u}) \equiv 0$, the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat.

4.1. Reciprocal transformations which preserve the flatness property of the metric and Lie-equivalent systems

We end the paper giving the geometrical interpretation of theorem 4.1 in the case in which both the initial and the transformed metrics are flat. Indeed, local Hamiltonian systems connected by canonical reciprocal transformations have nice geometrical properties as first observed by Ferapontov [11]. Using the theorems proven by Ferapontov in [11] and theorem 4.1, in theorem 4.12 we show that the local Hamiltonian structures of two DN Hamiltonian systems in Riemann invariants are connected by a canonical reciprocal transformation if and only if the associated hypersurfaces are Lie equivalent.

A DN hydrodynamic-type system as in (1) in flat coordinates takes the form

$$u^i_t = v^i_j(\boldsymbol{u})u^i_x = \epsilon^i \delta^{ij} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta H}{\delta u^j}\right), \tag{51}$$

with $\epsilon^i = \pm 1$ and the Hamiltonian $H = \int h(\boldsymbol{u})\,\mathrm{d}x$. To each system as in (51), there corresponds a hypersurface M^n in a pseudoeuclidean space E^{n+1} in such a way that equations (51) may be transformed into the form

$$\boldsymbol{n}_t = \boldsymbol{r}_x, \tag{52}$$

where $\boldsymbol{n}$ and $\boldsymbol{r}$ are respectively the unit normal and the radius vector of M^n (see [11]). Let $u^1, \ldots, u^n$ be any system of curvilinear coordinates on M^n. Since the tangent bundle TM^n is spanned by $\frac{\partial \boldsymbol{r}}{\partial u^i}$, $i = 1, \ldots, n$ and $\frac{\partial \boldsymbol{n}}{\partial u^i} \in TM^n$, $i = 1, \ldots, n$, it is possible to introduce the so-called Weingarten (or shape) operator $w^i_j(\boldsymbol{u})$, by the formulae

$$\frac{\partial \boldsymbol{n}}{\partial u^j} = w^i_j(\boldsymbol{u}) \frac{\partial \boldsymbol{r}}{\partial u^j},$$

and (52) may be rewritten in the form (51), with $v^i_j = \left(w^i_j\right)^{-1}$. Then the eigenvalues of the velocities $v^i_j(\boldsymbol{u})$ are the radii of the principal curvatures of M^n and the corresponding eigenfoliations are the curvature surfaces of M^n (see [11]). In particular, the hypersurface M^n is called Dupin if its principal curvatures are constant along the corresponding curvature hypersurfaces and such hypersurfaces correspond to weakly nonlinear hydrodynamic-type systems (i.e. each eigenvalue of the matrix $v^{ij}(\boldsymbol{u})$ in (51) is constant along the corresponding eigenfolation) as proven in [11].

Following [11], let us call the hypersurfaces associated with two DN systems as in (51) Lie equivalent if they are connected by a Lie sphere transformation (see [3, 14]).

The $n + 2$-canonical integrals (the n Casimirs, the momentum and the Hamiltonian) take the following form in the flat coordinates $u^1, \ldots, u^n$ (see [11]):

$$H = h\,\mathrm{d}x + \frac{1}{2}\left(\sum_{m=1}^{n} \epsilon^m (\partial_m h)^2 + 1\right)\mathrm{d}t,$$

$$P = \frac{1}{2}\left(\sum_{m=1}^{n} \epsilon^m u_m^2 + 1\right)\mathrm{d}x - \left(h - \sum_{m=1}^{n} u_m \partial_m h\right)\mathrm{d}t,$$

$$U^i = u^i\,\mathrm{d}x + \epsilon^i \partial_i h\,\mathrm{d}t, \qquad i = 1, \ldots, n.$$

Then the following theorem settles the following important relation between equivalent hypersurfaces and reciprocal transformations.

Theorem 4.11. *[11]*

(A) Suppose that the associated hypersurfaces of two DN systems as in (51) are Lie equivalent. Then the local Hamiltonian structures of the systems themselves are connected by a reciprocal transformation.

(B) Suppose that the local Hamiltonian structures of two DN systems are connected by the canonical reciprocal transformation

$$\mathrm{d}\hat{x} = \alpha H + \beta P + \sum_{m=1}^{n} \gamma_i U^i + \eta_1\,\mathrm{d}x + \eta_2\,\mathrm{d}t,$$
$$\mathrm{d}\hat{t} = \tilde{\alpha} H + \beta P + \sum_{m=1}^{n} \tilde{\gamma}_i U^i + \tilde{\eta}_1\,\mathrm{d}x + \tilde{\eta}_2\,\mathrm{d}t,$$

with $\alpha, \beta, \gamma_m, \eta_j, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}_m, \tilde{\eta}_j, (m = 1, \dots, n,\ j = 1, 2)$ *constants such that*

$$\begin{aligned}
&(\alpha+\eta_1)^2 + (\beta+\eta_2)^2 - \sum_{m=1}^{n} \epsilon^m \gamma_m^2 - \eta_1^2 - \eta_2^2 = 0,\\
&(\tilde{\alpha}+\tilde{\eta}_1)^2 + (\tilde{\beta}+\tilde{\eta}_2)^2 - \sum_{m=1}^{n} \epsilon^m \tilde{\gamma}_m^2 - \tilde{\eta}_1^2 - \tilde{\eta}_2^2 = 0,\\
&(\tilde{\alpha}+\tilde{\eta}_1)(\alpha+\eta_1) + (\tilde{\beta}+\tilde{\eta}_2)(\beta+\eta_2) - \sum_{m=1}^{n} \epsilon^m \tilde{\gamma}_m\gamma_m - \tilde{\eta}_1\eta_1 - \tilde{\eta}_2\eta_2 = 0.
\end{aligned} \tag{53}$$

Then the hypersurfaces associated with the two DN systems are Lie equivalent.

We recall that any $n \times n$ DN-type system as in (51) admits the $n + 2$-canonical integrals, so that theorem 4.11 applies also to the case in which Riemann invariants do not exist.

If we restrict ourselves to the case of DN systems which possess Riemann invariants, then the compatibility conditions (53) in the flat coordinates have their correspondence in conditions (A.i)–(A.iii) expressed in the Riemann invariants in theorem 4.1.

Moreover, the same theorem gives the complete characterization of the reciprocal transformations which preserve local Hamiltonian structure when Riemann invariants exist, so that the following stronger geometrical characterization holds in the present case.

Theorem 4.12. *Let* $n \geqslant 5$. *The hypersurfaces associated with two diagonalizable strictly hyperbolic DN systems are connected by a Lie sphere transformation if and only if the corresponding local Hamiltonian structures of the two DN systems are connected by canonical reciprocal transformation satisfying theorem 4.1.*

Finally, we would like to point out that there is no geometrical interpretation of the reciprocal transformations when the locality of the Hamiltonian structure is not preserved by the transformation and both the initial and the transformed systems are of DN type. The most interesting example in this class are the genus g modulated Camassa–Holm equations already mentioned in the introduction: such a system possesses two compatible flat metrics which are mapped to two non-flat metrics associated with the g modulated equations of the first negative Korteweg–de Vries flow by a reciprocal transformation as proven in [2]. Then, from theorem 4.11, it follows that the hypersurfaces associated with the two systems are not Lie equivalent.

Acknowledgments

This research has been partially supported by ESF Programme MISGAM, by RTN ENIGMA and by PRIN2006 'Metodi geometrici nella teoria delle onde non lineari ed applicazioni'. I warmly thank F V Ferapontov, M Pavlov and Y Zhang for their interest in the present research. I am also grateful to the referees for many useful remarks and especially to the second referee for many valuable observations which helped me in improving the manuscript.

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